Whitham-Toda Hierarchy

And

N = 2 Supersymmetric Yang-Mills Theory

Toshio NAKATSU [1] and Kanehisa TAKASAKI [2][†]

[1]*Department of Mathematics and Physics, Ritsumeikan University*
Kusatsu,Shiga 525-77,Japan
[2]*Department of Fundamental Sciences, Faculty of Integrated Human Studies, Kyoto University*
Yoshida-Nihonmatsu-cho,Sakyo-ku,Kyoto 606,Japan

ABSTRACT

The exact solution of $N = 2$ supersymmetric $SU(N)$ Yang-Mills theory is studied in the framework of the Whitham hierarchies. The solution is identified with a homogeneous solution of a Whitham hierarchy. This integrable hierarchy (Whitham-Toda hierarchy) describes modulation of a quasi-periodic solution of the (generalized) Toda lattice hierarchy associated with the hyperelliptic curves over the quantum moduli space. The relation between the holomorphic pre-potential of the low energy effective action and the τ function of the (generalized) Toda lattice hierarchy is also clarified.

† E-mail: takasaki@yukawa.kyoto-u.ac.jp, nakatsu@tkyvax.phys.s.u-tokyo.ac.jp

arXiv:hep-th/9509162v1 28 Sep 1995

Recently Seiberg and Witten [1] obtained exact expressions for the metric on the moduli space and the dyon spectrum of $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory by using a version of the Olive-Montonen duality [2] and holomorphy [3] of $4d$ supersymmetric theories. Their approach has been generalized to the case of other Lie group [4],[5]. Especially surprising in these results is unexpected emergence of elliptic (or hyperelliptic) curves and their periods. Although these objects appear in the course of determining the holomorphic pre-potential $\mathcal{F}$ of the exact low energy effective actions, physical significance of the curves themselves is unclear yet. It will be important to clarify their physical role. An interesting step in this direction has been taken [6] from the view of integrable systems, in which the correspondence between the Seiberg-Witten solution [1] and the Gurevich-Pitaevsky solution [7] to the elliptic Whitham-KdV equation [8] is pointed out.

In this article we shall consider the exact solution of $N = 2$ supersymmetric $SU(N)$ Yang-Mills theory in the framework of the Whitham hierarchies [9] and show that it can be identified with a homogeneous solution of a Whitham hierarchy. This Whitham hierarchy (Whitham-Toda hierarchy) turns out to be modulation equations [10] of a quasi-periodic solution of the (generalized) Toda lattice hierarchy associated with the hyperelliptic curves over the moduli space of Yang-Mills theory. In particular the relation between the pre-potential of this supersymmetric theory and the τ function of the (generalized) Toda lattice hierarchy will be clarified.

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One of the main ingredients in the analysis [1], [4] of the exact solution of $N = 2$ supersymmetric $SU(N)$ Yang-Mills theory is the meromorphic differential dS

$$dS = \frac{x\frac{dP(x)}{dx}}{y}dx \tag{1}$$

on the family of the hyperelliptic curves

$$C: \ y^2 = P(x)^2 - \Lambda^{2N} \ , \qquad P(x) = x^N + \sum_{k=0}^{N-2} u_{N-k}x^k. \tag{2}$$

Each u_k is an order parameter of $N = 2$ supersymmetric $SU(N)$ Yang-Mills theory. $u = (u_2, \cdots, u_N)$ become the parameters of the flat moduli. Λ is the lambda-parameter of this

gauge theory and we henceforth fix its value. The spectrum of excitations in the theory will be measured by the units

$$a_i = \oint_{\alpha_i} dS \quad , \quad a_{D,i} = \oint_{\beta_i} dS \quad (1 \le i \le N-1), \tag{3}$$

where α_i and $\beta_i (1 \le i \le N-1)$ are the standard symplectic basis of homology cycles of the hyperelliptic curve C. In particular each α_i is a cycle which encircles counterclockwise the cut between two neighboring branch points in $x-$plane. Let p_∞ and $\tilde{p}_\infty$ be the two points at infinity of the Riemann sheets of $C : x(p_\infty) = x(\tilde{p}_\infty) = \infty$. Since the pole divisor of dS (1) is $2p_\infty + 2\tilde{p}_\infty$ and it has no residues at these points, one can decompose it into the sum :

$$dS = dX_{\infty,1} + \sum_{i=1}^{N-1} a_i dz_i \quad , \tag{4}$$

where $dz_i(1 \le i \le N-1)$ is the holomorphic differential normalized by the condition $\oint_{\alpha_i} dz_j = \delta_{i,j}$. $dX_{\infty,1}$ is the meromorphic differential of second kind (its second order poles are at p_∞ and $\tilde{p}_\infty$) with vanishing periods along the α-cycles, that is, $\oint_{\alpha_i} dX_{\infty,1} = 0$ for $\forall i$.

Let us investigate the role of dS from the view of integrable system. For this purpose we introduce the following meromorphic functions h and $\tilde{h}$ on the curve C,

$$h = y + P(x) \quad , \quad \tilde{h} = -y + P(x) \quad , \tag{5}$$

and then consider the effect of infinitesimal deformation of the moduli parameters u with h (or $\bar{h}$) being fixed. Under this condition one can obtain

$$\frac{\partial}{\partial u_{N-k}} dS \bigg|_{fix\ h} = -\frac{x^k}{y} dx \quad . \tag{6}$$

After changing the moduli parameters from u to $a = (a_1, \cdots, a_{N-1})$ the above equation reads

$$\frac{\partial}{\partial a_i} dS \bigg|_{fix\ h} = dz_i \quad , \tag{7}$$

which implis that dS satisfies the following system of differential equations:

$$\frac{\partial}{\partial a_i} dz_j = \frac{\partial}{\partial a_j} dz_i \quad (1 \le i, j \le N-1), \tag{8}$$

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where the derivation by the moduli parameters a is understood to be partial derivation fixing h and the other a's. Differential equations of this form (8) were first derived by Flashka, Forest and MacLaughlin [11] as modulation equations of quasi-periodic solutions in soliton theory. The concept of modulated quasi-periodic solutions originates in Whitham's work on the KdV equation [8], and because of this, this type of systems are called Whitham equations. Remarkably, Whitham equations themselves are integrable systems. In particular, as soliton equations admit infinitely many commuting flows (which constitute an integrable hierarchy), the associated Whitham equations also have infinitely many extra commuting flows [9]. These commuting flows are generated by a set of meromorphic differentials $\{d\Omega_A\}_{A \in I}$, where flows are labeled by indices $A, B, \cdots$, and the hierarchy can be written as integrability conditions of those differentials. In the above case, these equations are given by

$$\frac{\partial}{\partial a_i} dz_j = \frac{\partial}{\partial a_j} dz_i \quad , \quad \frac{\partial}{\partial T_A} dz_i = \frac{\partial}{\partial a_i} d\Omega_A \quad , \quad \frac{\partial}{\partial T_A} d\Omega_B = \frac{\partial}{\partial T_B} d\Omega_A \quad , \tag{9}$$

where, as in (8), the derivation by the time variables mean derivation fixing h.

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Now we specify the meromorphic differentials $d\Omega_A$. They consist of two types of meromorphic differentials $d\Omega_{\infty,n}$ and $d\tilde{\Omega}_{\infty,n}$ of second-kind ($n \geq 1$): $d\Omega_{\infty,n}$ ($d\tilde{\Omega}_{\infty,n}$) has a pole of order $n+1$ at p_∞ ($\tilde{p}_\infty$) and is holomorphic elsewhere. We also introduce a meromorphic differential $d\Omega_{\infty,0} (\equiv d\tilde{\Omega}_{\infty,0})$ of third-kind : $d\Omega_{\infty,0}$ has simple poles at p_∞ and $\tilde{p}_\infty$ with $res_{p_\infty} d\Omega_{\infty,0} = -res_{\tilde{p}_\infty} d\Omega_{\infty,0} = 1$ and is holomorphic elsewhere. All these differentials are normalized such that they have no periods along any α_i-cycle ; $\oint_{\alpha_i} d\Omega_{\infty,n} = \oint_{\alpha_i} d\tilde{\Omega}_{\infty,n} = 0$ for $\forall i$, and determined by h and $\tilde{h}$ through the following prescription. Let us define local coordinates z_∞ and $\tilde{z}_\infty$ in neighborhoods of p_∞ and $\tilde{p}_\infty$ as

$$z_\infty^N = h^{-1} \quad , \quad \tilde{z}_\infty^N = \tilde{h}^{-1} \quad . \tag{10}$$

Because the divisors of h and $\tilde{h}$ are respectively $N\tilde{p}_\infty - Np_\infty$ and $Np_\infty - N\tilde{p}_\infty$ [1] it follows that $z_\infty(p_\infty) = 0$ and $\tilde{z}_\infty(\tilde{p}_\infty) = 0$. In a neighborhood of p_∞, they can be written

$$d\Omega_{\infty,n} = \{-nz_\infty^{-n-1} - \sum_{m \geq 1} q_{m,n} z_\infty^{m-1}\} dz_\infty \quad (n \geq 1),$$

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[1] This follows from the relation: $h\tilde{h} = \Lambda^{2N}$.

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$$d\tilde{\Omega}_{\infty,n} \;=\; \{\delta_{n,0}z_\infty^{-1} - \sum_{m\geq 1} r_{m,n}z_\infty^{m-1}\}dz_\infty \qquad (\, n \geq 0 \,), \tag{11}$$

while, in a neighborhood of $\tilde{p}_\infty$,

$$d\Omega_{\infty,n} \;=\; \{-\delta_{n,0}\tilde{z}_\infty^{-1} - \sum_{m\geq 1} \bar{r}_{n,m}\tilde{z}_\infty^{m-1}\}d\tilde{z}_\infty \qquad (\, n \geq 0 \,),$$

$$d\tilde{\Omega}_{\infty,n} \;=\; \{-n\tilde{z}_\infty^{-n-1} - \sum_{m\geq 1} \bar{q}_{m,n}\tilde{z}_\infty^{m-1}\}d\tilde{z}_\infty \qquad (\, n \geq 1 \,). \tag{12}$$

Integrability conditions (9) clearly ensure the existence of a differential dS which satisfies

$$\frac{\partial}{\partial a_i}dS = dz_i, \quad \frac{\partial}{\partial T_n}dS = d\Omega_{\infty,n}, \quad \frac{\partial}{\partial \bar{T}_n}dS = d\tilde{\Omega}_{\infty,n}, \quad \frac{\partial}{\partial T_0}dS = d\Omega_{\infty,0}, \tag{13}$$

where $1 \leq i \leq N-1$ and $n \geq 1$. We denote time variables of the flows generated by $d\Omega_{\infty,n}$, $d\Omega_{\infty,0}$ and $d\tilde{\Omega}_{\infty,n}$ as T_n, T_0 and $\bar{T}_n$ respectively. Now let us construct a function $F(a,T,\overline{T})$ from this differential dS by the following equations

$$\frac{\partial F}{\partial a_i} = \frac{1}{2\pi\sqrt{-1}}\oint_{\beta_i} dS \quad (\equiv \frac{a_{D,i}}{2\pi\sqrt{-1}}\,),$$

$$\frac{\partial F}{\partial T_n} = -res_{p_\infty}z_\infty^{-n}dS\,, \qquad \frac{\partial F}{\partial \bar{T}_n} = -res_{\tilde{p}_\infty}\tilde{z}_\infty^{-n}dS\,,$$

$$\frac{\partial F}{\partial T_0} = -res_{p_\infty}\ln z_\infty dS + res_{\tilde{p}_\infty}\ln \tilde{z}_\infty dS\,, \tag{14}$$

where $1 \leq i \leq N-1$ and $n \geq 1$. Notice that consistency of this definition of the F-function can be checked by using integrability conditions (9) and the Riemann bilinear relation for meromorphic differentials [12]. As an example let us prove the equality:

$$\frac{\partial}{\partial T_n}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{\beta_i} dS\right) = \frac{\partial}{\partial a_i}\left(-res_{p_\infty}z_\infty^{-n}dS\right) \qquad (n \geq 1). \tag{15}$$

In fact, by setting $\Omega_{\infty,n}(p) = \int^p d\Omega_{\infty,n}$, the L.H.S of eq.(15) can be transformed as

$$\frac{\partial}{\partial T_n}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{\beta_i} dS\right) \;=\; \frac{1}{2\pi\sqrt{-1}}\oint_{\beta_i} d\Omega_{\infty,n}$$

$$=\; \frac{-1}{2\pi\sqrt{-1}}\sum_{j=1}^{N-1}\left\{\oint_{\alpha_j} d\Omega_{\infty,n}\oint_{\beta_j} dz_i - \oint_{\alpha_j} dz_i \oint_{\beta_j} d\Omega_{\infty,n}\right\}$$

$$=\; -res_{p_\infty}\Omega_{\infty,n}dz_i$$

$$=\; -res_{p_\infty}z_\infty^{-n}dz_i,$$

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which is equal to the R.H.S of eq.(15). We also notice that, owing to definition (14), the local behaviors of dS can be described by the F-function as

$$dS = \left\{ -\sum_{n\geq 1} nT_n z_\infty^{-n-1} + T_0 z_\infty^{-1} - \sum_{n\geq 1} \frac{\partial F}{\partial T_n} z_\infty^{n-1} \right\} dz_\infty \qquad \text{around } p_\infty,$$

$$dS = \left\{ -\sum_{n\geq 1} n\bar{T}_n \tilde{z}_\infty^{-n-1} - T_0 \tilde{z}_\infty^{-1} - \sum_{n\geq 1} \frac{\partial F}{\partial \bar{T}_n} \tilde{z}_\infty^{n-1} \right\} d\tilde{z}_\infty \qquad \text{around } \tilde{p}_\infty. \qquad (16)$$

An interesting class of solutions of the Whitham hierarchy are solutions that enjoy the homogeneity condition

$$\sum_{i=1}^{N-1} a_i \frac{\partial F}{\partial a_i} + \sum_{n\geq 0} T_n \frac{\partial F}{\partial T_n} + \sum_{n\geq 1} \bar{T}_n \frac{\partial F}{\partial \bar{T}_n} = 2F. \qquad (17)$$

In this case, dS, which is introduced by eq.(13), has a form analogous to (4)

$$dS = \sum_{i=1}^{N-1} a_i dz_i + \sum_{n\geq 0} T_n d\Omega_{\infty,n} + \sum_{n\geq 1} \bar{T}_n d\tilde{\Omega}_{\infty,n}. \qquad (18)$$

Therefore one can reproduce dS of (1) by simply setting $T_1 = -\bar{T}_1 = 1$ and other T_A-variables zero. The pre-potential $\mathcal{F}$ of $N = 2$ supersymmetric $SU(N)$ Yang-Mills theory is now given by $\mathcal{F} = 2\pi\sqrt{-1}F$. Let us prove that (18) is consistent with (16). We first notice that eqs. (13) and (14) make it possible to express the coeffcents in (11) and (12) in terms of F. As for the holomorphic differentials dz_i let us write their local expansion as

$$dz_i = -\sum_{m\geq 1} \sigma_{i,m} z_\infty^{m-1} dz_\infty \qquad \text{around } p_\infty,$$

$$dz_i = -\sum_{m\geq 1} \bar{\sigma}_{i,m} \tilde{z}_\infty^{m-1} d\tilde{z}_\infty \qquad \text{around } \tilde{p}_\infty, \qquad (19)$$

where $1 \leq i \leq N-1$. Then, by using eqs.(13) and (14), we can also express the coefficients in (19) in terms of F. Inserting these local expansions into the R.H.S of (18), and recalling the homogeneity of F (17), one can reproduce (16).

For this homogeneous solution, we can write the F-function in terms of the β-periods of the differentials dz_i, $d\Omega_{\infty,n}$ and $d\Omega_{\infty,n}$ as follows. First notice that by homogeneity relation (17) and definition of F-function (14), F can be written as a sum of periods and residues of dS

$$F = \frac{1}{2} \left(\sum_{i=1}^{N-1} a_i \frac{\partial F}{\partial a_i} + \sum_{n\geq 0} T_n \frac{\partial F}{\partial T_n} + \sum_{n\geq 1} \bar{T}_n \frac{\partial F}{\partial \bar{T}_n} \right)$$

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$$= \frac{1}{2} \left\{ \sum_{i=1}^{N-1} \frac{a_i}{2\pi\sqrt{-1}} \oint_{\beta_i} dS - \sum_{n\geq1} T_n res_{p_\infty} z_\infty^{-n} dS \right.$$

$$\left. - \sum_{n\geq1} \tilde{T}_n res_{\tilde{p}_\infty} \tilde{z}_\infty^{-n} dS - T_0 \left(res_{p_\infty} \ln z_\infty dS - res_{\tilde{p}_\infty} \ln \tilde{z}_\infty dS \right) \right\}. \tag{20}$$

Since dS has the form given in (18), this expression of F can be evaluated further to become the aforementioned form. Eventually, we obtain

$$\begin{aligned} F &= \frac{1}{4\pi\sqrt{-1}} \sum_{i,j=1}^{N-1} \tau_{i,j} a_i a_j + \sum_{i=1}^{N-1} a_i \left\{ \sum_{k\geq1} \left(\sigma_{i,k} T_k + \bar{\sigma}_{i,k} \bar{T}_k \right) + \bar{\sigma}_{i,0} T_0 \right\} \\ &+ \frac{1}{2} \sum_{k,l\geq1} q_{k,l} T_k T_l + \sum_{k,l\geq1} r_{k,l} T_k \bar{T}_l + \frac{1}{2} \sum_{k,l\geq1} \bar{q}_{k,l} \bar{T}_k \bar{T}_l + \frac{1}{2} \bar{r}_{0,0} T_0^2 \\ &+ T_0 \sum_{k\geq1} \left(r_{k,0} T_k + \bar{r}_{0,k} \bar{T}_k \right), \end{aligned} \tag{21}$$

where the quantities $\tau_{i,j}, \bar{\sigma}_{i,0}$ and $\bar{r}_{0,0}$ are introduced as

$$\begin{aligned} \tau_{i,j} &= \oint_{\beta_i} dz_j \qquad (1 \leq i, j \leq N-1), \\ \bar{\sigma}_{i,0} &= \frac{1}{2\pi\sqrt{-1}} \oint_{\beta_i} d\Omega_{\infty,0} \qquad (1 \leq i \leq N-1), \\ \bar{r}_{0,0} &= -res_{p_\infty} \ln z_\infty d\Omega_{\infty,0} + res_{\tilde{p}_\infty} \ln \tilde{z}_\infty d\Omega_{\infty,0}, \end{aligned} \tag{22}$$

and other quantities in (21) are those appearing in expansions (11),(12) and (19).

3

Now we will discuss the relation between this homogeneous solution of the Whitham hierarchy (9) and the Toda lattice hierarchy. Let us denote Toda lattice time variables by $t = (t_1, t_2, \cdots)$, $\bar{t} = (\bar{t}_1, \bar{t}_2, \cdots)$ and n ($n \in Z$). It is well-known [13] that the Toda lattice hierarchy has a quasi-periodic solution associated with a hyperelliptic curve C as in (2). In fact, this is a dimensionally reduced solution — in the lowest sector of the hierarchy, the solution gives an N-periodic solution of the Toda chain rather than the two-dimensional Toda field equations [2]. Since the flows of the Toda lattice hierarchy do not change this curve itself, the moduli parameters u are invariants (integrals of motion) of these solutions of the Toda lattice hierarchy.

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[2]This dimensional reduction is irrelevant in our discussion.

We introduce an additional new time variables $\theta = (\theta_1, \cdots, \theta_{N-1})$ into this solution. This is achieved by modifying the associated Baker-Akhiezer function [13] as

$$
\begin{aligned}
&\Psi(p; t, \bar{t}, n, \theta) \\
&= \exp\left\{-n \int^p d\Omega_{\infty,0} + \sum_{n \geq 1} t_n \int^p d\Omega_{\infty,n} + \sum_{n \geq 1} \bar{t}_n \int^p d\tilde{\Omega}_{\infty,n} + \sqrt{-1} \sum_{i=1}^{N-1} \theta_i \int^p dz_i\right\} \\
&\times \frac{\vartheta\left(z(p) - z(D) + \Delta - n(z(p_\infty) - z(\tilde{p}_\infty)) + \sum_{n \geq 1} t_n \sigma_n + \sum_{n \geq 1} \bar{t}_n \bar{\sigma}_n + \frac{1}{2\pi} \sum_{i=1}^{N-1} \theta_i \tau_i\right)}{\vartheta\left(z(p_\infty) - z(D) + \Delta - n(z(p_\infty) - z(\tilde{p}_\infty)) + \sum_{n \geq 1} t_n \sigma_n + \sum_{n \geq 1} \bar{t}_n \bar{\sigma}_n + \frac{1}{2\pi} \sum_{i=1}^{N-1} \theta_i \tau_i\right)} \\
&\times \frac{\vartheta\left(z(p_\infty) - z(D) + \Delta\right)}{\vartheta\left(z(p) - z(D) + \Delta\right)}.
\end{aligned}
\tag{23}
$$

"z" is the Abel mapping, that is, $z(p) =^T (z_1(p), \cdots, z_{N-1}(p))$ where $z_i(p) = \int^p dz_i$. D is a positive divisor of degree $N - 1$ and Δ is the Riemann constant. σ_l, $\bar{\sigma}_l$ ($l \geq 1$) and τ_i ($1 \leq i \leq N - 1$) are $N - 1$ dimensional complex vectors whose components are given by $\sigma_l =^T (\sigma_{l,1}, \cdots, \sigma_{l,N-1})$, $\bar{\sigma}_l =^T (\bar{\sigma}_{l,1}, \cdots, \bar{\sigma}_{l,N-1})$, and $\tau_i =^T (\tau_{i,1}, \cdots, \tau_{i,N-1})$. Notice that Baker-Akhiezer function Ψ (23) reduces to that of the ordinary Toda lattice as $\theta \to 0$. For non-zero values of θ, Ψ is quasi-periodic along the β-cycles,

$$
\Psi(p; t, \bar{t}, n, \theta) \overset{\beta_i}{\longmapsto} e^{\sqrt{-1}\theta_i} \Psi(p; t, \bar{t}, n, \theta),
\tag{24}
$$

and periodic along the α-cycles. Hence the Baker-Akhiezer function Ψ can also be regarded as a section of a flat line bundle over C now labeled by θ-variables. We also notice that this solution of the (generalized) Toda lattice is N-periodic (with respect to n). One can check the following periodicity at the level of Baker-Akhiezer function

$$
\Psi(p; t, \bar{t}, n + N, \theta) = \text{const.} \; h(p) \; \Psi(p; t, \bar{t}, n, \theta).
\tag{25}
$$

One may introduce the τ-function of this solution following the prescription of Toda lattice hierarchy [14]. In particular the local expansions of Ψ around p_∞ and $\tilde{p}_\infty$ will be described by the τ-function : In a neighborhood of p_∞

$$
\Psi(p; t, \bar{t}, n, \theta) = z_\infty^{-n} e^{\sum_{l \geq 1} t_l z_\infty^{-l}} \frac{\tau(t - [z_\infty], \bar{t}, n, \theta)}{\tau(t, \bar{t}, n, \theta)},
\tag{26}
$$

and in a neighborhood of $\tilde{p}_\infty$

$$
\Psi(p; t, \bar{t}, n, \theta) = \tilde{z}_\infty^n e^{\sum_{l \geq 1} \bar{t}_l \tilde{z}_\infty^{-l}} \frac{\tau(t, \bar{t} - [\tilde{z}_\infty], n + 1, \theta)}{\tau(t, \bar{t}, n, \theta)},
\tag{27}
$$

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where $[z_\infty] = (\frac{z_\infty}{1}, \frac{z_\infty^2}{2}, \frac{z_\infty^3}{3}, \cdots)$.

By matching this expression of Ψ with (23), we can write down the τ-function in terms of the theta function and the coefficients of local expansion of dz_i, etc. We thus obtain the following expression of the τ function :

$$
\begin{aligned}
&\tau(t, \bar{t}, n, \theta) \\
&= e^{\hat{F}(t,\bar{t},n.\theta)} \\
&\quad \times \vartheta \left(-(n-1)z(p_\infty) + nz(\tilde{p}_\infty) - z(D) + \Delta + \sum_{n\geq 1} t_n \sigma_n + \sum_{n\geq 1} \bar{t}_n \bar{\sigma}_n + \frac{1}{2\pi} \sum_{i=1}^{N-1} \theta_i \tau_i \right),
\end{aligned}
\tag{28}
$$

where $\hat{F}$ is a polynomial of second degree in $t, \bar{t}, n$ and θ given by [3]

$$
\begin{aligned}
\hat{F}(t, \bar{t}, n, \theta) &= \frac{1}{2} \sum_{k,l\geq 1} q_{k,l} t_k t_l + \sum_{k,l\geq 1} r_{k,l} t_k \bar{t}_l + \frac{1}{2} \sum_{k,l\geq 1} \bar{q}_{k,l} \bar{t}_k \bar{t}_l - \frac{1}{4\pi\sqrt{-1}} \sum_{i,j=1}^{N-1} \tau_{i,j} \theta_i \theta_j \\
&\quad + \frac{n(n-1)}{2} \bar{r}_{0,0} + \sqrt{-1} \sum_{i=1}^{N-1} \theta_i \left\{ \sum_{k\geq 1} (\sigma_{i,k} t_k + \bar{\sigma}_{i,k} \bar{t}_k) - n \bar{\sigma}_{i,0} \right\} \\
&\quad - n \sum_{k\geq 1} r_{k,0} t_k - (n-1) \bar{r}_{0,k} \bar{t}_k + \sum_{k\geq 1} d_k t_k + \sum_{k\geq 1} \bar{d}_k \bar{t}_k + \bar{d}_0 n.
\end{aligned}
\tag{29}
$$

Note that the quadratic part of $\hat{F}$ (29) has almost the same form as F in (21). This is not accidental. In fact it turns out that the Whitham hierarchy in the previous section gives modulation equations of the aforementioned quasi-periodic solution of the (generalized) Toda lattice hierarchy. Let us explain this relation in more detail following Bloch and Kodama [15] [4]. For this purpose we introduce "slow" time variables [5] $T_l, \bar{T}_l (l \geq 1), T_0$ and $a_i (1 \leq i \leq N-1)$ by

$$
T_l = \epsilon t_l , \quad \bar{T}_l = \epsilon \bar{t}_l , \quad T_0 = -\epsilon n , \quad a_i = \sqrt{-1}\epsilon \theta_i.
\tag{30}
$$

and consider the asymptotics of Baker-Akhiezer function (23) and τ-function (28) as $\epsilon \to 0$. In the slow time variables, they turn out to have the following expression

$$
d \ln \Psi \left(p; T/\epsilon, \bar{T}/\epsilon, -T_0/\epsilon, a/\sqrt{-1}\epsilon \right) = \epsilon^{-1} \sum_{n\geq 0} \epsilon^n dS^{(n)}(p; T, \bar{T}, T_0, a),
$$

---

[3]In (29) $d_k (k \geq 1)$ and $\bar{d}_k (k \geq 0)$ are the constants irrelevant to our discussion and we do not describe their explicit forms.

[4]Their Whitham hierarchy for the Toda lattice is different from ours, but the idea of deriving modulation equations is the same.

[5]ϵ is arbitary small. So these variables are small relative to the (generalized) Toda time variables.

$$\ln \tau \left(T/\epsilon, \bar{T}/\epsilon, -T_0/\epsilon, a/\sqrt{-1}\epsilon \right) \;=\; \epsilon^{-2} \sum_{n \geq 0} \epsilon^n F^{(n)}(T, \bar{T}, T_0, a). \tag{31}$$

The leading-order terms $dS^{(0)}$ and $F^{(0)}$ are given by the same expressions as dS in (18) and F in (21) respectively. We now allow the moduli parameters u to depend on the slow variables,

$$u_k = u_k(T, \bar{T}, T_0, a). \tag{32}$$

In other words, the hyperelliptic curve C can now "slowly" vary as $C = C(T, \bar{T}, T_0, a)$. We further require that this "modulated" quasi-periodic wave be still a solution of the (generalized) Toda lattice hierarchy. This induces a system of differential equations (modulation equations) to the moduli parameters. By the theory of Whitham hierarchies [9], [10], [11], these modulation equations turn out to be nothing but our Whitham hierarchy (9).

4

So far, we have considered the integrable structure of $N = 2$ supersymmetric Yang-Mills theory by embedding the system into the Whitham hierarchy (9). The hierarchy is constructed by adding the additional T- and $\bar{T}$-flows to the flows generated by the holomorphic differentials. Meanwhile, the time variables a of the latter flows, along with their dual variables a_D, constitute the units of spectrum in the theory. Though all these flows can be identified with those of the (generalized) Toda lattice, physical roles of the additional ones are not clear. In this final section we discuss their physical implication. In particular we show that the solution of the Whitham hierarchy that we have considered satisfies the same Virasoro constraints as appear in topological string theory.

For this purpose we define the following function Q by using the differential dS introduced in (13) ,

$$Q = \frac{dS}{dh}. \tag{33}$$

Since $S(p) = \int^p dS$ is multi-valued, Q can also be multi-valued in general. Actually, it is known [9] that a class of solutions of the Whitham hierarchy are characterized by geometric conditions concerning single-valuedness and regularity of Q. For an example

one can determine a solution of the Whitham hierarchy by the condition that Q be single-valued and have no singular points other than p_∞ and $\tilde{p}_\infty$. This is indeed the solution that we have considered in the preceding sections. Let us now impose a stronger condition that this single-valued Q be regular at $\tilde{p}_\infty$, too. This regularity condition forces $\forall \bar{T}_k = 0$. Furthermore, by the single-valuedness, it follows that

$$res_{p_\infty} h^{n+1} Q dS = 0 \tag{34}$$

for $\forall n \geq -1$. This is a consequence of the Riemann bilinear relation. Since we already know how the differentials $h^{n+1} Q dS$ behave around p_∞, eqs.(34) can be rephrased as constraints on the F-function. They are given by

$$\sum_{k \geq 1} k T_k \frac{\partial F}{\partial T_{k+nN}} + \frac{1}{2} \sum_{k+l=nN} \frac{\partial F}{\partial T_k} \frac{\partial F}{\partial T_l} + \frac{1}{2} \sum_{k+l=N} k l T_k T_l \delta_{n+1,0} = 0 \tag{35}$$

for $\forall n \geq -1$. Notice that we can make F independent of $T_{kN} (\forall k \geq 1)$ in (35) without any loss of generality. This is because the corresponding generators become exact forms,

$$d\Omega_{\infty,kN} = dh^k \quad . \tag{36}$$

Hence constraints (35) now become precisely same as those on the free energy of the A_{N-1} topological string (in spherical limit) [16],[17]. In this topological string theory, time variables such as T_n play the role of coupling constants of the chiral primary fields or their gravitational descendants.

From this observation it would be a very fascinating idea to interpret the meromorphic differentials $d\Omega_{\infty,n}$ ($n \geq 1$) as the chiral primary fields or their gravitational descendants of the A_{N-1} topological string. It should be noticed that one can also give the same interpretation on the differentials $d\tilde{\Omega}_{\infty,k}(k \geq 1)$. So it might be possible to interpret N=2 supersymmetric Yang-Mills theory as a coupled system of two topological string models. As we have mentioned above, the corresponding solution of the Whitham hierarchy is characterized by the condition that Q be single-valued and have singularities at most at p_∞ and $\tilde{p}_\infty$. Again by the Riemann bilinear relations, this condition can be restated as

$$res_{p_\infty} h^{n+1} Q dS + res_{\tilde{p}_\infty} h^{n+1} Q dS = 0 \tag{37}$$

for $\forall n$. We can rewrite (37) as constraints on the F-function. They turn out to be

$$\sum_{k \geq 1} k T_k \frac{\partial F}{\partial T_{k+nN}} + \frac{1}{2} \sum_{k+l=nN} \frac{\partial F}{\partial T_k} \frac{\partial F}{\partial T_l} = \Lambda^{2nN} \left\{ \sum_{k \geq 1} (k+nN) \bar{T}_{k+nN} \frac{\partial F}{\partial \bar{T}_k} + \frac{1}{2} \sum_{k+l=nN} k l \bar{T}_k \bar{T}_l \right\},$$

$$\sum_{k\geq 1} k\bar{T}_k \frac{\partial F}{\partial \bar{T}_{k+nN}} + \frac{1}{2} \sum_{k+l=nN} \frac{\partial F}{\partial \bar{T}_k} \frac{\partial F}{\partial \bar{T}_l} = \Lambda^{2nN} \left\{ \sum_{k\geq 1} (k+nN) T_{k+nN} \frac{\partial F}{\partial T_k} + \frac{1}{2} \sum_{k+l=nN} kl T_k T_l \right\},$$
(38)

where $n \geq 0$.

4.1 Acknowledgements

The second author (K.T.) would like to thank A. Gorsky, S. Kharchev, A. Marshakov, A. Mironov, and A. Morozov for fruitful discussions. The idea to connect the hyperelliptic cirve C with the Toda lattice was inspired by this Moscow group.

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